TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 340, Number 2, December 1993

NORMALITY IN X^2 FOR COMPACT X

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ABSTRACT. In 1977, the second author announced the following consistent negative answer to a question of Katětov: Assuming MA+$\neg$CH, there is a compact nonmetric space X such that X^2 is hereditarily normal. We give the details of this example, and construct another example assuming CH. We show that both examples can be constructed so that $X^2\backslash\Delta$ is perfectly normal. We also construct in ZFC a compact nonperfectly normal X such that $X^2\backslash\Delta$ is normal.

1. INTRODUCTION

In his classical paper [K], Katětov showed that if X and Y are infinite compact spaces and $X \times Y$ is hereditarily normal, then X and Y are perfectly normal. By Šneĭder's theorem that a compact space with a G_δ-diagonal is metrizable [S], Katětov concludes that if X is compact and X^3 is hereditarily normal, then X is metrizable. He asked if the same conclusion could be obtained assuming only that X^2 is hereditarily normal. In 1977, the second author obtained a counterexample assuming Martin's Axiom plus the negation of the Continuum Hypothesis (MA + $\neg$CH). This result was announced in [Ny_1], and an outline of the proof appeared there, although with many details omitted. A complete proof appears in this paper for the first time. We also construct a (necessarily different, as will be seen) counterexample assuming CH.

Since any counterexample must be perfectly normal, it is probably not surprising that our examples are related to Alexandrov's double arrow space $D = [0, 1] \times \{0, 1\}$ with the lexicographic order topology, for it is in some sense the only known example in ZFC of a compact perfectly normal nonmetrizable space. (See [G_1] for a discussion of this.) The double arrow space has also been called the "split interval" because one can think of obtaining it by splitting each $x \in [0, 1]$ into two points x^- and x^+, and putting the order topology on these points, where x^- is declared to be less than x^+ and otherwise the order is the natural one inherited by $[0, 1]$. Now if $A \subset [0, 1]$, let $D(A)$ be the same as above but with only the points of A split; of course $D(A)$ is just the quotient space of D obtained by identifying x^- with x^+ for all $x \notin A$. If A is uncountable, then $D(A)$ is a compact perfectly normal nonmetrizable space.

Received by the editors December 11, 1989 and, in revised form, January 28, 1991.
1991 *Mathematics Subject Classification*. Primary 54A35, 54D15, 54D30.
Research of the first author partially supported by NSF DMS-8503328.
Research of the second author partially supported by NSF DMS-8703008.

0002-9947/93 \$1.00 + \$.25 per page

The counterexample to Katětov's question under $MA + \neg CH$ announced in [Ny_1] is $D(A)$, where $\aleph_1 \leq |A| < 2^{\aleph_0}$. We show in fact that a counterexample of the form $D(A)$ exists assuming only the existence of an uncountable Q-set (which is weaker than assuming $MA + \neg CH$), i.e., an uncountable subspace of the reals in which every subset is a relative G_δ-set. Jones' lemma [J] shows that no $D(A)$ can be a counterexample under CH (or $2^{\aleph_0} < 2^{\aleph_1}$), because $D(A)^2$ is separable and contains the discrete subspace $\{\langle a^-, a^+\rangle : a \in A\}$. However, we show that a modification of the split interval topology essentially due to M. Wage [W] can be used to construct a counterexample under CH.

It is still unknown if there is a ZFC counterexample to Katětov's question. The second author also announced the following in [Ny_1]:

Theorem. *Assuming* $2^{\aleph_0} < 2^{\aleph_1}$, *if* X *is compact and* X^2 *is hereditarily normal, then either* (a) X *is an* L*-space, i.e., hereditarily Lindelöf but not separable*; (b) X^2 *is an* S*-space, i.e., hereditarily separable but not Lindelöf*; *or* (c) X^2 *contains both an* S*-space and an* L*-space.*

Recall that S. Todorcevic [To_1] has constructed a model of set theory in which there are no S-spaces, though $2^{\aleph_0} = 2^{\aleph_1}$ in this model. It is unknown if there is a model with no L-spaces, or whether there is a model with $2^{\aleph_0} < 2^{\aleph_1}$ and no S- or L-spaces (yielding a model in which Katětov's question has a positive answer). See §6 for more discussion of this, and a proof of the above theorem, with the set-theoretic hypothesis weakened to "there are no uncountable Q-sets". Our CH example has the property that X^2 is an S-space.

The second author also showed [Ny_2] that under $MA + \neg CH$, a compact space X is metrizable if X^2 is hereditarily collectionwise normal (CWN), or hereditarily normal and hereditarily collectionwise Hausdorff (CWH). We show that "X^2 is hereditarily strongly CWH" can be added to the list. Our CH example has all these properties but is not metrizable, so these results are independent of ZFC. On the other hand, the first author showed [G_2] in ZFC that a compact X is metrizable if X^2 is hereditarily paracompact, or just if $X^2 \backslash \Delta$ is paracompact, where Δ is the diagonal.

In a personal communication P. Kombarov asked the first author the following related questions:

(1) If X is compact and $X^2 \backslash \Delta$ is perfectly normal, must X be metrizable?

(2) If X is compact and $X^2 \backslash \Delta$ is normal, must X be perfectly normal?

Question (1) is closely related to the following question which appears in [P_3]:

(3) If X and Y are compact and $X \times Y$ is perfectly normal, must at least one of X and Y be metrizable?

It is not difficult to show that a counterexample to (1) must contain counterexamples to (3) (otherwise, one shows that the supposed counterexample to (1) is locally metrizable, hence metrizable). Rudin [Ru] shows that assuming axiom $\diamond$ (which holds in Gödel's constructible universe L) there are two Souslin lines whose product is perfectly normal. But a Souslin line will not do for (1) because Rudin showed in the same paper that $X^2 \backslash \Delta$ is never hereditarily normal when X is a Souslin line. However, a later example of Todorcevic [To_2] contains the key idea. He shows that if A and B are disjoint subsets of $[0, 1]$ such that there does not exist an uncountable one-to-one monotone function from any uncountable subset of A to B, then $D(A) \times D(B)$ is perfectly normal. Now it is consistent with MA$+\neg$CH that there is an uncountable $E \subset [0, 1]$ such that every one-to-one monotone function from an uncountable subset of E to E has a fixed point (see [AS]; such a set E is called "2-entangled"). We show that $D(E)^2$ is a counterexample to (1) under MA $+ \neg$CH. We also show that our CH counterexample to Katětov's question can be constructed so that $X^2\backslash\Delta$ is perfectly normal. This leaves the following open problem:

Problem. Is there in ZFC a compact nonmetrizable space X such that (a) $X^2\backslash\Delta$ is perfectly normal or (b) $X \times Y$ is perfectly normal for some compact nonmetrizable space Y?

Regarding question (2), it is shown in [AK] that if X is compact and $X^2\backslash\Delta$ is normal, then X must be first countable. We show however that the answer to (2) is "no"; the CH example above can be modified to obtain in ZFC a compact space X such that $X^2\backslash\Delta$ is normal, even CWN, yet X is not hereditarily normal.

2. The MA examples

To show that $D(A)$ is a counterexample to Katětov's question, what is needed precisely is that A is uncountable, and A^2 is both a Q-set and a λ'-set in $[0, 1]^2$, i.e., A^2 is a Q-set and N is a G_δ-set in $A^2 \cup N$ for every countable $N \subset [0, 1]^2$. (It is equivalent to say that $A^2 \cup N$ is a Q-set for every countable $N \subset [0, 1]^2$.) Under MA $+ \neg$CH, every uncountable $A \subset [0, 1]$ of size less than $2^{\aleph_0}$ has these properties. It turns out that the existence of such a set follows from the existence of an uncountable Q-set—see §5 for a proof.

As suggested in [P$_3$], the following result of T. Przymusinski is a useful ingredient in our proof (but it is by no means the only nor the most important one).

Theorem 2.1 [P$_1$]. *If $Y \subset R^2$ is a Q-set, Y is normal as a subspace of the Sorgenfrey plane.*

Theorem 2.2. *Let $A \subset [0, 1]$ be such that A^2 is a Q-set and a λ'-set. Then $D(A)^2$ is hereditarily normal.*

Before proving this theorem, we state a few preliminary lemmas and establish some notation.

Fix a subset A of $[0, 1]$ and, for convenience, denote $D(A)$ by P.

There is a natural quotient map π from P to I, sending a^+ and a^- to a for each $a \in A$, and another from P^2 to I^2. Images under these two maps will be designated by asterisks, e.g. given $q \in P^2$, the image of q will be denoted q^*. Inverse images will be designated by the symbol $^\#$, but $^{*\#}$ will be abbreviated $^\#$. For instance: if $q \in (I\backslash A)^{2\#}$, then $q^* \in (I\backslash A)^2$ and $q^\# = \{q\}$; if $q \in A^\#$, then $q^\#$ is a two-element set. The authors have found it helpful to think of P^2 and I^2 with almost the same pictures, thinking of the points of each $q^\#$ as superimposed on one another, hence the choice of a notation that does not distinguish too strongly between objects in the respective spaces.

The set A is replaced in P by the two disjoint sets A^+ and A^- [obvious definition]. Each a^+ has a local base in P consisting of the topologically open intervals

$$I_n(a^+) = [a^+, a^+ + 1/n) = \pi^{-1}[a, a + 1/n) - \{a^-\}.$$

We similarly define $I_n(a^-)$ as $\pi^{-1}[a-1/n, a)-\{a^+\}$, while if $b \in P - A^\#$ we let $I_n(b)$ as $\pi^{-1}(b-1/n, b+1/n)$. Then for any point $q = \langle x, y\rangle \in P^2$ [we use $\langle\ ,\ \rangle$ for ordered pairs] we let $B_n(q) = I_n(x) \times I_n(y)$.

Lemma 2.3. *If A is a Q-set, then $P^2 - (A^\#)^2$ is hereditarily Lindelöf.*

Proof. Of course, $P^2-(A^\#)^2 = (P-A^\#)^2 \cup [(P-A^\#)\times A^\#] \cup [A^\# \times (P-A^\#)]$. Now $(P-A^\#)^2$ is metrizable, and $(P-A^\#)\times A^\#$ is the disjoint union of $(P-A^\#)\times A^+$ and $(P-A^\#)\times A^-$; of course, $A^\# \times (P-A^\#)$ splits up similarly. Since A^+ and A^- are homeomorphic to subsets of the Sorgenfrey line, they are hereditarily Lindelöf [E, 3.8.14: the proof goes through for subspaces]. Then $(P-A^\#)\times A^+$ is the product of a second countable space and a hereditarily Lindelöf space, so it too is hereditarily Lindelöf. The other pieces of $P^2 - (A^\#)^2$ are similarly hereditarily Lindelöf. Finally, observe that the countable union of hereditarily Lindelöf spaces is likewise hereditarily Lindelöf. □

Corollary 2.4. *If A^2 is a Q-set, then P^2 is the union of five perfectly normal subspaces.*

Proof. Recall that a space is perfectly normal iff it is normal and every open subset is an F_σ. By Lemma 2.3, $P^2 - (A^\#)^2$ is perfectly normal (as is every hereditarily Lindelöf regular space, [E, 3.8A]). In their relative topologies, $(A^+)^2$, $(A^-)^2$, $A^+ \times A^-$, and $A^- \times A^+$ are clearly homeomorphic to subspaces of the Sorgenfrey plane and each other, and so by Theorem 2.1 they are normal. In [HM] it is shown that every closed subset of the Sorgenfrey plane is a G_δ, and this property clearly carries over to subspaces. □

Despite Corollary 2.4, P^2 is not perfectly normal by Šneĭder's theorem if A is uncountable. To simplify the proof of its hereditary normality when A^2 is a Q-set and a λ'-set, we will take A to be dense in I; the general case follows from the elementary fact [E, p. 96] that hereditary normality is preserved by closed maps, and the fact (which will be shown in 5.1) that $(A \cup N)^2$ is also a Q-set and a λ'-set if N is a countable dense subset of $[0, 1]$. Indeed, if A is not dense, we can add a countable dense subset of I to A, and there is a natural (closed) quotient map from the resulting P^2 to the one using A through re-identifying points that were split in the added set.

We omit the easy proof of

Lemma 2.5. *Let A be dense in* $[0, 1]$. *Then $(A^+)^2$ is dense in P^2, and if U is relatively open in $(A^+)^2$, then $U \subset \overline{U}^\circ$.* [*Closures and interiors are taken in P^2 unless otherwise indicated.*]

Our proof of Theorem 2.2 will also use

Lemma 2.6 [E, proof of 1.5.14]. *If H and K are subsets of a space X, then there exist disjoint open sets U and V of X such that $H \subset U$, $K \subset V$ if, and only if, there is a countable cover of $H \cap K$ by open sets, each of whose closures meets at most one of H, K.*

Proof of Theorem 2.2. Recall that sets H and K are said to be *mutually separated* in a space if $\overline{H} \cap K = \overline{K} \cap H = \varnothing$, and that hereditary normality is equivalent to *complete normality*, the property that mutually separated sets can be put into disjoint open sets [E, 2.1.7].

Let H and K be mutually separated in P^2. Using Lemma 2.3 and regularity of P^2, we can get $H \cap (A^\#)^2$ into a countable union of open sets whose closure misses K.

Now we need to show that there exist countably many open sets covering $H \cap (A^+)^2 = H_1$, $H \cap (A^- \times A^+) = H_2$, $H \cap (A^-)^2 = H_3$, and $H \cap (A^+ \times A^-) = H_4$ whose closures miss K. The notation H_i is suggested by trigonometry (e.g. the points of H_1 have "first quadrantic neighborhoods") and similarly we define K_i for $i = 1, 2, 3, 4$. We carry out the argument for H_1; that for the other H_i merely involves changes in geometric orientation.

By Theorem 2.1, there is a relatively open subset V of $(A^+)^2$ containing H_1 whose closure misses K_1, and by Lemma 2.5 there is an open subset W of P^2 with the same properties.

For each $q \in H_1$, let $n(q)$ be the least n such that $B_n(q)$ is contained in W and its closure misses K. let $H_1^m = \{q | n(q) = m\}$ and let each H_1^m be expressed as a countable union of relatively closed subspaces of $(A^+)^2$. Reindexing, we get $H_1 = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed subspace of $(A^+)^2$ and for each $q \in F_n$, $n(q) \leq n$.

Fix F_n, and let I^2 be covered by countably many open squares S, each $< 1/n$ on a side. For each S, let

$$U(S) = \bigcup \{B_n(q) | q \in S^\# \cap F_n\} \cap S^\# .$$

This is an open subspace of P^2 containing $F_n \cap S^\#$, and our proof will be complete after we cover $F_n \cap S^\#$ with countably many open subsets of $U(S)$ each of whose closures miss K.

Down in I^2, let

$$\partial = (\text{boundary of } U(S)^*) \cap S .$$

This is an arc which is the union of the closures of the graphs of two functions:

$$f(x) = \inf\{y | \langle x, y\rangle \in U(S)^*\}, \quad g(y) = \inf\{x | \langle x, y\rangle \in U(S)^*\} .$$

These are nonincreasing, because if $x_1 < x_2$ the inf is taken over a smaller set for x_1; similarly when $y_1 < y_2$. Indeed, $\langle x, y\rangle \in U(S)^*$ iff there exists $\langle x', y'\rangle \in F_n^*$ with $x' \leq x$ and $y' \leq y$. There is a natural total order on ∂, from left to right and top to bottom. Horizontal and vertical segments of ∂ are at discontinuity points of f and g, of which there can be at most countably many. Let Q be the set of all endpoints of such segments.

Still in I^2, give each point in the interior of $U(S)^*$ [= all points of S above or to the right of ∂] an open neighborhood whose closure is contained in S and pick a countable subcover of this interior. Their preimages in P^2 will take care of $F_n \backslash \partial^\#$.

Not all points of $\partial^\#$ need be in $\overline{U(S)}$; in fact, any horizontal segment of ∂ along the line $y = a \in A$ will have only the points whose second coordinate is a^+ in $\overline{U(S)}$; the ones whose second coordinate is a^- have neighborhoods "facing downward." Similarly, if an interval of a vertical line $x = a$ is in ∂ then only the "right half," $x = a^+$, will be in $\overline{U(S)}$. Moreover, no point of $(A^-)^2$ can be in $\partial^\# \cap \overline{U(S)}$ because its basic neighborhoods face away from $\partial^\#$, as in the case of p_3 in Figure 1. Hence $\overline{U(S)}$ misses K_3 altogether, and it also misses K_1 since it is a subset of $\overline{W}$.

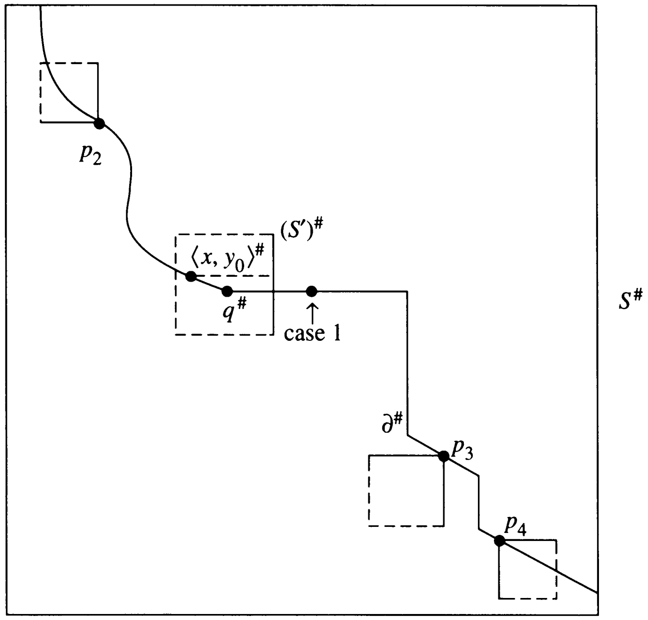

FIGURE 1

Claim 1. Points of $K \cap \overline{U(S)}$ can only occur along horizontal and/or vertical segments of $\partial^{\#}$.

Indeed, having ruled out K_1 and K_3, we are left with points of K_2 and K_4, like p_2 and p_4 in Figure 1, and points of $K \cap [P^2 - (A^{\#})^2]$. If a point p of $K_2 \cup K_4$ is not along some horizontal or vertical segment, then it represents a "point of decrease" of both f and g, but then because of the direction in which the neighborhoods of p face, this would put p in the closure of F_m, giving a contradiction. Since points of $K \cap [P^2 - (A^{\#})^2]$ have "larger" neighborhoods, this reasoning applies to them also.

Claim 2. Let $D = Q \cup (F_n^* \cap \partial)$. Let $F_n \cap \partial^{\#} = \bigcup_{m=1}^{\infty} C_m$ where each C_m^* is closed in the relative (Euclidean) topology of D. For each m, let

$$U(S)_m = \bigcup \{B_n(q) | q \in C_m\} \cap S^{\#}.$$

Then the closure of any $U(S)_m$ misses K.

Once the claim is proven, Theorem 2.2 follows by Lemma 2.6.

Write $U(S)_m = U$. Since $U \subset U(S)$, it is enough to exhibit a neighborhood of each point p of $K \cap \overline{U(S)}$ which misses U. By Claim 1, p^* can only lie on a horizontal and/or vertical segment of ∂.

Case 1. *p^* is on a horizontal segment but is not the right endpoint.* Let q^* be the left endpoint of this segment. There cannot be a point of F_n^* anywhere between q^* and p^* (except possibly p^* itself), otherwise p would be in $U(S)$ if it is in $\overline{U(S)}$. In particular, q^* is not in F_n^*, so there is an open square S' centered on q^* missing C_m^* and also p^*. Let $\langle x, y_0\rangle$ be a point of S' on ∂ preceding q^*, so that y_0 is greater than the second coordinate of q^* (otherwise the left endpoint status of q^* would be violated). The portion R_0 of S' below the line $y = y_0$ can be extended arbitrarily far to the left and down without encountering C_m^*; thus R_0 misses $U(S)_m^*$. We extend $R_0^{\#}$ to the right to form a rectangular open set R in P^2 which picks up p but no points of C_m. [It may be necessary to have the right edge of R split $p^\#$.] This R is the desired neighborhood.

Case 2. *p^* is on a vertical segment but is not the upper endpoint.* Let q^* be the lower endpoint of this segment. Argue as above, defining $\langle x_0, y\rangle$ to follow q^* on ∂, letting R_0 be the portion of S' to the left of $x = x_0$ extending $R_0^\#$ upward to pick up p.

Case 3. *p^* is a right endpoint but not a vertical endpoint.* Then p is in $P \times A^+$, otherwise it would be in the closure of R_m. Since p is not in K_1 it is either in K_2, whence the argument in Case 1 applies, or in $(P - A) \times A^+$, when (let $p = \langle x, a^+\rangle$) we can chop off the rectangle $R_0^\#$ of Case 1 right between $y = a^-$ and $y = a^+$, and then extend it beyond p to the right edge of $S^\#$ without encountering any point of C_m.

Case 4. *p^* is both an upper endpoint and a right endpoint.* If p is in K_2 this can be handled like Case 1. If p is in K_4 it can be handled like Case 2. If p were in K_1 or K_3 it could not be in $\overline{U(S)}$. Other possibilities for p can be handled by extending rectangles for both versions of q^* as in the previous cases. □

To get a space $D(A)$ as in Theorem 2.2 with the further property that $D(A)^2\backslash\Delta$ is perfectly normal, we will need to assume more about the set A.

Definition 2.3. An uncountable subset A of R is *2-entangled* if there is no uncountable monotone function f from a subset of A to A with no fixed points.

Shelah showed in [AS] that the existence of a 2-entangled set is consistent with $\mathrm{MA} + \neg\mathrm{CH}$. Thus it is consistent that there exists a 2-entangled set $A \subset [0, 1]$ such that A^2 is a Q-set and a λ'-set, and this is precisely what we need for the example.

Theorem 2.4. *Suppose $A \subset [0, 1]$ is 2-entangled, and A^2 is a Q-set and a λ'-set. Then $D(A)^2\backslash\Delta$ is perfectly normal.*

Proof. Assume $A \subset [0, 1]$ is 2-entangled. We will show that every closed subset of $D(A)^2\backslash\Delta$ is a G_δ-set; the rest follows from Theorem 2.2.

As in the previous example, let $\pi: D(A) \to [0, 1]$ be the natural projection. Further, if $x \in [0, 1]$ and $e \in \{-, +\}$, we use x^e to denote the obvious if $x \in A$, else $x^e = x$; also, we use $-e$ to denote the other member of $\{-, +\}$. In the remainder of the proof, the letters e and f will always denote members of $\{-, +\}$.

We will use the basic fact that any closed subset K of $D(A)$ "splits" at most countably many $x \in [0, 1]$, i.e.

$$\{x \in [0, 1]: \exists e(x^e \in K, x^{-e} \notin K)\}$$

is countable. (This follows from the fact that any uncountable set of reals clusters to some point of itself from both the left and right, and it is how one shows that the double arrow space is perfectly normal.)

Denote the space $D(A)^2\backslash\Delta$ by X, and suppose H is a closed subset of X. We will prove that H is a G_δ-set in X.

Note that if $\langle x^e, y^f\rangle \notin H$, and either $\langle x^e, y^{-f}\rangle \notin H$ or $\langle x^{-e}, y^f\rangle \notin H$, then $\langle x^e, y^f\rangle$ is a member of a product $\pi^{-1}(B) \times U$ or $U \times \pi^{-1}(B)$ which misses H, where B is a member of some countable base $\mathscr{B}$ for $[0, 1]$, and U is open in $D(A)$. Let O be the union of all such products which miss H. Using the facts that $\mathscr{B}$ is countable and $D(A)$ is hereditarily Lindelöf, we see that O is an open F_σ-set which misses H.

Let $H' = X\backslash O$. Then H' is a closed G_δ-set containing H, and if $\langle x^e, y^f\rangle \in H'\backslash H$, then either

(i) $x = y$ and $e \neq f$; or

(ii) $x \neq y$, $\langle x^e, y^{-f}\rangle \in H$, and $\langle x^{-e}, y^f\rangle \in H$.

Let $F = \{\langle x^e, y^f\rangle \in H'\backslash H : x \neq y\}$.

Claim. $|F| \leq \omega$.

Suppose $|F| > \omega$. Note that for fixed x (or fixed y), the set

$$\{\langle x^e, y^f\rangle \in H'\backslash H : x \neq y\}$$

is countable, for otherwise there are fixed x, e, and f and uncountably many y's such that $\langle x^e, y^f\rangle \in H$ but $\langle x^e, y^{-f}\rangle \notin H$, contradicting the fact that a closed subset of $D(A)$ splits at most countably many $y \in [0, 1]$.

Now we can find fixed e, f, and $\langle x_\alpha, y_\alpha\rangle$ for $\alpha < \omega_1$, such that $x_\alpha \neq y_\alpha$, $\langle x_\alpha^e, y_\alpha^f\rangle \in H'\backslash H$, and $\{x_\alpha, y_\alpha\} \cap \{x_\beta, y_\beta\} = \varnothing$ whenever $\beta \neq \alpha$.

Let us assume $e = f = +$, and show that this leads to a contradiction. It will be clear that a contradiction can also be obtained for any other values of e and f.

For each $\alpha < \omega_1$, choose rationals $q_\alpha > x_\alpha$ and $r_\alpha > y_\alpha$ such that

$$[x_\alpha^+, q_\alpha^+) \times [y_\alpha^+, r_\alpha^+) \subset X\backslash H.$$

(The intervals are taken using the natural order on $D(A)$.) There exist rationals q and r, and an uncountable $W \subset \omega_1$ such that $q_\alpha = q$ and $r_\alpha = r$ for all $\alpha \in W$. The map $x_\alpha \to y_\alpha$, $\alpha \in W$, is one-to-one with no fixed points, so since A is 2-entangled, it cannot be order-reversing. Hence for some $\beta \neq \alpha \in W$, $x_\alpha < x_\beta$ and $y_\alpha < y_\beta$. But then $\pi^{-1}(x_\beta) \times \pi^{-1}(y_\beta) \subset X\backslash H$, contradicting $\langle x_\beta^e, y_\beta^f\rangle \in H'$ (by (ii) above). This completes the proof of the claim.

It is easy to check that

$$D = \{\langle x^e, x^{-e}\rangle : x \in A, e \in \{-, +\}\}$$

is closed discrete in X. Thus

$$H'\backslash H = F \cup (D \cap (H'\backslash H))$$

is an F_σ-set, so H is G_δ in H' and also in X. □

3. The CH example

Assuming CH, no $D(A)$ with A uncountable can have either the property that $D(A)^2$ is hereditarily normal or that $D(A)^2\backslash\Delta$ is perfectly normal: $\{(x^+, x^-) : x \in A\}$ is a closed discrete set of size continuum in the separable space $D(A)^2\backslash\Delta$, so $D(A)^2\backslash\Delta$ cannot be normal by Jones' lemma. In this section we show that in any case CH does imply the existence of a compact nonmetrizable space X satisfying the conditions of Theorem 2.4.

Theorem 3.1. (CH) *There is a compact nonmetrizable space* X *such that* X^2 *is hereditarily normal and hereditarily separable, and* $X^2\backslash\Delta$ *is perfectly normal.*

Proof. The construction of this example is a modification of a technique due to M. Wage [W], which in turn is a modification of the so-called "Kunen line" construction [JKR].

Let C be the Cantor set, and let L be a dense Luzin subset of C, that is, $L \cap N$ is countable for each nowhere dense subset N of C. (Such sets exist assuming CH.) Let L^0 and L^1 be disjoint copies of L. If $x \in L$, let x^0 and x^1 denote the copies of x in L^0 and L^1, respectively; for convenience, if $x \in C\backslash L$, we let $x^0 = x^1 = x$.

The underlying set for our space is $X = (C\backslash L) \cup L^0 \cup L^1$. There is a natural projection $\pi\colon X \to C$ which will turn out to be continuous.

Let $\mathscr{B}$ be a countable clopen base for C, and let τ_0 be the compact zero-dimensional topology on X generated by $\{\pi^{-1}(B)\colon B \in \mathscr{B}\}$.

Let $\{x_\alpha\colon \alpha < \omega_1\}$ be an enumeration of C. For $\alpha < \omega_1$, let $C_\alpha = \{x_\beta\colon \beta < \alpha\}$; we may assume $C_\omega \subset L$ and is dense in C. Let $\{A_\alpha\colon \omega \le \alpha < \omega_1\}$ index all countable subsets of C^2 whose elements are disjoint as unordered pairs; arrange this indexing so that $A_\alpha \subset C_\alpha^2$ for each α.

We define the topology on X by inductively defining neighborhood bases of x_α^0 and x_α^1, for $x_\alpha \in L$. To start, give the points x_n^0 and x_n^1, $n < \omega$, usual double arrow neighborhoods.

Inductively define, for $\omega \le \alpha < \omega_1$, $\beta \le \alpha$, and $\sigma \in 2^2$, sequences

$$\{(x_\alpha(n, \sigma), y_\alpha(n, \sigma))\colon n < \omega\} \subset C_\alpha^2,$$

and sets $Y(\beta, \alpha, \sigma) \subset C_\alpha$ satisfying:

(i) For each $\sigma \in 2^2$, the sequence $\{x_\alpha(n, \sigma)\colon n < \omega\}$ converges to x_α, and for $\sigma \neq \sigma'$, $\{x_\alpha(n, \sigma)\colon n \in \omega\} \cap \{x_\alpha(n, \sigma')\colon n \in \omega\} = \varnothing$;

(ii) $Y(\beta, \alpha, \sigma) = \{y_\alpha(n, \sigma)\colon (x_\alpha(n, \sigma), y_\alpha(n, \sigma)) \in A_\beta\}$;

(iii) $\{y\colon (x_\alpha, y) \in \overline{A}_\beta\} \subset \overline{Y(\beta, \alpha, \sigma)}$;

(iv) If $\beta \le \delta < \alpha$ and $x_\alpha \in \overline{Y(\beta, \delta, \sigma')}$, then $Y(\beta, \delta, \sigma') \cap \{x_\alpha(n, \sigma)\colon n \in \omega\}$ is infinite for each σ, $\sigma' \in 2^2$;

(v) If $\beta < \alpha$ and there are $(x_n, y_n) \in A_\beta$ with $x_n \neq y_n$ and $(x_n, y_n) \to (x_\alpha, x_\alpha)$, then $Y(\beta, \alpha, \sigma') \cap \{x_\alpha(n, \sigma)\colon n \in \omega\}$ is infinite for each σ, $\sigma' \in 2^2$.

We show how to define the $x_\alpha(n, \sigma)$'s given that the $x_\beta(n, \sigma)$'s have been defined for all $\beta < \alpha$. Let

$$\{G_{\sigma k \beta}\colon \sigma \in 2^2, k < \omega, \beta < \alpha\} \cup \{H_{\sigma\beta\delta\tau}\colon \sigma, \tau \in 2^2, \beta, \delta < \alpha\} \\ \cup \{I_{\sigma\tau\beta}\colon \sigma, \tau \in 2^2, \beta < \alpha\}$$

be a partition of the even natural numbers into infinite sets. Choose a sequence $\{(x_n(\beta), y_n(\beta)\}_{n<\omega} \subset A_\beta$ converging to (x_α, x_α) with $x_n(\beta) \neq y_n(\beta)$ for all $n \in \omega$, if such a sequence exists. Let $\{z_k(\beta)\colon k < \omega\}$ be a countable dense subset of $\overline{A}_\beta \cap (\{x_\alpha\} \times C)$. Let d denote the usual Euclidean distance on C.

Suppose $n \in \omega$ is even and $x_\alpha(i, \sigma)$ has been defined for each $i < n$ and $\sigma \in 2^2$.

Choose eight points

$$\{x_\alpha(i, \sigma)\colon \sigma \in 2^2, i \in \{n, n+1\}\}$$

within $1/2^n$ of x_α which are distinct from each other and the $x_\alpha(i, \sigma)$'s, $i < n$, and satisfying whichever of the following conditions apply:

(a) if $n \in G_{\sigma k \beta}$ and $\overline{A}_\beta \cap (\{x_\alpha\} \times C) \neq \varnothing$, then for some (unique) y, $(x_\alpha(n, \sigma), y) \in A_\beta$ and $d(y, z_k(\beta)) < 1/2^n$;

(b) if $n \in H_{\sigma\beta\delta\tau}$ and $x_\alpha \in \overline{Y(\beta, \delta, \tau)}$, then $x_\alpha(n, \sigma) \in \overline{Y(\beta, \delta, \tau)}$;

(c) if $n \in I_{\sigma\tau\beta}$ and $\{(x_m(\beta), y_m(\beta))\}_{m<\omega}$ has been defined, then $(x_\alpha(n, \sigma), x_\alpha(n+1, \tau)) = (x_m(\beta), y_m(\beta))$ for some $m < \omega$.

It is possible to choose the $x_\alpha(i, \sigma)$'s in this way because at stage n there are points satisfying whichever of (a)–(c) applies arbitrarily close to x_α, while only finitely many points have been defined so far. It is elementary to check that (i)–(v) hold.

Suppose we have defined neighborhoods of x^0_β and x^1_β for $x_\beta \in L$, $\beta < \alpha$, where $\omega \leq \alpha < \omega_1$, such that the topology generated by τ_0 and these neighborhoods are a zero-dimensional compact topology τ_α, and τ_α is Hausdorff with respect to $\{x^0_\beta, x^1_\beta\}$, $\beta < \alpha$.

If $n \in \omega$, $\sigma \in 2^2$, and $x_\alpha(n, \sigma) \notin L$, choose a τ_α-clopen set $B(n, \sigma)$ containing $x_\alpha(n, \sigma)$, while if $x_\alpha(n, \sigma) \in L$ choose τ_α-clopen sets $B(n, \sigma, e)$, $e \in 2$, containing $x_\alpha(n, \sigma)^e$ in such a way that the collection of all $B(n, \sigma)$'s and $B(n, \sigma, e)$'s is pairwise disjoint, and x^0_α and x^1_α are the only τ_α-limit points.

Now define a basic $\tau_{\alpha+1}$-neighborhood of x^0_α to be x^0_α together with

$$\bigcup\{B(n, \sigma) \colon \sigma(0) = 0, n \geq k\}$$

and

$$\bigcup\{B(n, \sigma, e) \colon \sigma(e) = 0, n \geq k]$$

for $k \in \omega$; i.e., a basic neighborhood of x^0_α is x^0_α together with tails of the sequences $\{B(n, \sigma)\}_{n<\omega}$ for $\sigma \in 2^2$ with $\sigma(0) = 0$, and tails of the sequences $\{B(n, \sigma, e)\}_{n<\omega}$ for $\sigma \in 2^2$ with $\sigma(e) = 0$. A basic neighborhood of x^1_α is the complement of a basic neighborhood of x^0_α, intersected with a τ_0 neighborhood, i.e., with $\pi^{-1}(B)$ for some clopen $B \subset C$.

The desired topology on X is $\bigcup_{\alpha<\omega_1} \tau_\alpha$. Note that for $x \notin L$, $\{\pi^{-1}(B) \colon B$ clopen in $C, x \in B\}$ is a base at x.

Fact 1. If B_i is a neighborhood of x^i_α, then $B_0 \cup B_1 \supset \pi^{-1}(B)$ for some clopen $B \subset C$.

Proof. Clear from the definition of the topology.

Fact 2. X is a compact T_2-space.

Proof. Clearly X is a T_2-space. That X is compact follows immediately from Fact 1.

Fact 3. If $A \subset \omega$ is infinite, $\sigma \in 2^2$, and either $\sigma(0) = \sigma(1)$, or $x_\alpha(n, \sigma) \in L$ for each $n \in A$, then

$$\{x_\alpha(n, \sigma)^e\}_{n \in A} \to x^{\sigma(e)}_\alpha.$$

Proof. This follows immediately from the definition of $\tau_{\alpha+1}$, noting $x_\alpha(n, \sigma)^e \in B(n, \sigma, e)$ if $x_\alpha(n, \sigma) \in L$, $x_\alpha(n, \sigma) \in B(n, \sigma)$ if $x_\alpha(n, \sigma) \notin L$, and $\sigma(0) = \sigma(1)$ implies $\sigma(0) = 0$ iff $\sigma(e) = 0$ for some (all) $e < 2$.

Fact 4. Suppose $\gamma \geq \alpha > \beta$ and $(x_\alpha, x_\gamma) \in \overline{A_\beta}$. Let $e, e', f, f' \in 2$. Then

$$(x_\alpha^f, x_\gamma^{f'}) \in \overline{\{(a^e, b^{e'}) : (a, b) \in A_\beta\}}$$

if either

(a) $\gamma > \alpha$; or

(b) $\gamma = \alpha$, and $f = f'$ or $\forall (a, b) \in A_\beta (a^e \neq b^{e'})$.

Proof. (a) Choose $\sigma, \tau \in 2^2$ such that $\sigma(e) = \sigma(1-e) = f$ and $\tau(e') = \tau(1-e') = f'$. Since $(x_\alpha, x_\gamma) \in \overline{A_\beta}$, by property (iii) of the inductive construction of the $x_\alpha(n, \sigma)$'s, we have $x_\gamma \in \overline{Y(\beta, \alpha, \sigma)}$. By (iv), the set

$$A = \{n : x_\gamma(n, \tau) \in Y(\beta, \alpha, \sigma)\}$$

is infinite. Then $\{(x_\alpha(n, \sigma), (x_\gamma(n, \tau))\}_{n \in A} \subset A_\beta$ and by Fact 3,

$$\{(x_\alpha(n, \sigma)^e, x_\gamma(n, \tau)^{e'})\}_{n \in A} \to (x_\alpha^f, x_\gamma^{f'}).$$

(b) Assume $\gamma = \alpha$. Then $(x_\alpha, x_\alpha) \in \overline{A_\beta}$. If there are $(x_n, y_n) \in A_\beta$ with $x_n \neq y_n$ and $(x_n, y_n) \to (x_\alpha, x_\alpha)$, the proof is the same as (a) using condition (v) in place of (iv). So assume such (x_n, y_n) do not exist. Then by (iii),

$$A = \{n : (x_\alpha(n, \sigma), x_\alpha(n, \sigma)) \in A_\beta\}$$

is infinite. If for all $(a, b) \in A_\beta$, $a^e \neq b^{e'}$, then $e \neq e'$, and $a = b$ implies $a \in L$. So in this case or in the case $f = f'$, we may choose $\sigma \in 2^2$ with $\sigma(e) = f$ and $\sigma(e') = f'$. Then again by Fact 3,

$$\{(x_\alpha(n, \sigma)^e, (x_\alpha(n, \sigma)^{e'})\}_{n \in A} \to (x_\alpha^f, x_\alpha^{f'}).$$

Fact 5. Suppose Z is an uncountable subset of C^2 whose elements are disjoint as unordered pairs and $A \subset C^2$ is such that $Z \subset \overline{A}$. Then the following holds for all but countably many $(p, q) \in Z$: $\forall e, e', f, f' \in 2$,

$$(p^f, q^{f'}) \in \overline{\{(a^e, b^{e'}) : (a, b) \in A\}}$$

whenever

(a) $q \neq p$, or

(b) $q = p$, and $f = f'$ or $\forall (a, b) \in A (a^e \neq b^{e'})$.

Proof. Without loss of generality, Z is dense-in-itself in C^2. Let $\{E_n\}_{n<\omega}$ enumerate all members of some countable open base for C^2 meeting Z. Since $Z \cap E_n \subset \overline{A \cap E_n}$ and members of Z are disjoint as unordered pairs, we can inductively choose $e_n \in A \cap E_n \backslash \{e_i\}_{i<n}$ such that the e_n's are disjoint as unordered pairs. Then $E = \{e_n\}_{n<\omega} \subset A$ and $Z \subset \overline{E}$.

Let $E' = \{(b, a) : (a, b) \in E\}$. There exist β, $\beta' < \omega_1$ such that $E = A_\beta$ and $E' = A_{\beta'}$. Let $\alpha_0 > \beta \cup \beta'$, and let Z_0 be a countable subset of Z such that $(p, q) \in Z \backslash Z_0$ implies $\{p, q\} \cap C_{\alpha_0} = \varnothing$.

Let $(p, q) \in Z \backslash Z_0$ and $e, e', f, f' \in 2$. Suppose q does not appear before p in the indexing of C. By Fact 4, in either case 5(a) or 5(b), we have

$$(p^f, q^{f'}) \in \overline{\{(a^e, b^{e'}) : (a, b) \in A_\beta\}},$$

whence $(p^f, q^{f'}) \in \overline{\{(a^e, b^{e'}) : (a, b) \in A\}}$.

If q does come before p in the indexing of C, then by Fact 4 we have

$$(q^{f'}, p^f) \in \overline{\{(b^{e'}, a^e): (b, a) \in A_{\beta'}\}},$$

so again

$$(p^f, q^{f'}) \in \overline{\{(a^e, b^{e'}): (a, b) \in A\}}.$$

Fact 6. X is hereditarily separable.

Proof. Let $Y \subset X$ be uncountable. Without loss of generality, we prove that $Y_0 = \{y \in Y: y = \pi(y)^0\}$ is separable.

Let E be a countable dense subset of $\pi(Y_0)$, and let $A = \{(e, e): e \in E\}$. Then by Fact 5, for all but countably many $(\pi(y), \pi(y)) \in \pi(Y_0)$, we have

$$(\pi(y)^0, \pi(y)^0) \in \overline{\{(e^0, e^0): e \in E\}}.$$

It follows that Y_0 is separable.

Fact 7. X^2 is hereditarily separable.

Proof. If not, there exists an uncountable left-separated $Y = \{y_\alpha\}_{\alpha<\omega_1} \subset X^2$ (i.e., $y_\alpha \notin \overline{\{y_\beta\}}_{\beta<\alpha}$ for each $\alpha < \omega_1$). By Fact 6, we may assume that elements of $\pi(Y) \subset C^2$ are disjoint as unordered pairs (pass to an uncountable subset if necessary), and that $Y \subset X^2 \setminus \Delta$.

For each α, there are $e(\alpha)$, $f(\alpha) \in 2$ such that $y_\alpha = (p_\alpha^{e(\alpha)}, q_\alpha^{f(\alpha)})$ for some $(p_\alpha, q_\alpha) \in C^2$. For some $e, f \in 2$,

$$Y(e, f) = \{y_\alpha: e(\alpha) = e, f(\alpha) = f\}$$

is uncountable. Let A be a countable dense subset of $Z = \{(p_\alpha, q_\alpha): y_\alpha \in Y(e, f)\}$. Since $Y \subset X^2 \setminus \Delta$, $a^e \neq b^f$ for each $(a, b) \in A$. Since $Y(e, f)$ is left-separated, there are only countably many $(p, q) \in Z$ with

$$(p^e, q^f) \in \overline{\{(a^e, b^f): (a, b) \in A\}}.$$

This contradicts Fact 5.

Fact 8. Let $H \subset X$ be closed. Then $\{x \in C: \pi^{-1}(x) \cap H \neq \varnothing$ and $\pi^{-1}(x) \not\subset H\}$ is countable.

Proof. If not, then for some closed $H \subset X$ and $e \in \{0, 1\}$, $\{x: x^e \in H, x^{1-e} \notin H\}$ is uncountable. Let $Z = \{(x, x) \in C^2: x^e \in H, x^{1-e} \notin H\}$, and let A be a countable dense subset of Z. Since $x^{1-e} \notin H$ and $\overline{\{(a^e, a^e): (a, a) \in A\}} \subset \{(h, h): h \in H\}$, it follows that $(x^{1-e}, x^{1-e}) \notin \overline{\{(a^e, a^e): (a, a) \in A\}}$ for any $(x, x) \in Z$. This violates Fact 5.

Fact 9. X is perfectly normal.

Proof. Let H be closed in X. Then $\pi(H)$ is closed, hence a regular G_δ-set in C, so $\pi^{-1}(\pi(H))$ is a regular G_δ-set in X. By Fact 8, $|\pi^{-1}(\pi(H)) \setminus H| \leq \omega$. It follows that H is a regular G_δ-set in X.

Fact 10. If H is closed in $X^2 \setminus \Delta$, then the set $W = \{(x, y) \in C^2: (\pi^{-1}(x) \times \pi^{-1}(y)) \cap \overline{H} \neq \varnothing$ and $\pi^{-1}(x) \times \pi^{-1}(y) \not\subset \overline{H}\}$ is contained in $(C \times E) \cup (E \times C)$ for some countable $E \subset C$.

Proof. If not, we can find an uncountable $W' \subset W$ such that elements of W' are disjoint as unordered pairs. For each $(p, q) \in W'$, choose $g, g', f, f' \in 2$ with $(p^g, q^{g'}) \in \overline{H}$ and $(p^f, q^{f'}) \notin \overline{H}$. Choose also $e, e' \in 2$ with

$$(p^g, q^{g'}) \in \overline{\{(x^e, y^{e'}) : (x^e, y^{e'}) \in H\}}.$$

There is an uncountable set $Z \subset W'$ on which e, e', f, f' are fixed. Let A be countable dense in $\{(x, y) : (x^e, y^{e'}) \in H\}$. Then $Z \subset \overline{A}$, but

$$(p^f, q^{f'}) \notin \overline{\{(x^e, y^{e'}) : (x, y) \in A\}} \subset \overline{H}$$

for any $(p, q) \in Z$. Since $H \subset X^2\backslash\Delta$, $x^e \neq y^{e'}$ for all $(x, y) \in A$. Thus Fact 5 is violated.

Fact 11. $X^2\backslash\Delta$ is perfectly normal.

Proof. Let $H \subset X^2\backslash\Delta$ be closed. We will show that $(X^2\backslash\Delta)\backslash H$ can be written as a countable union of relatively clopen sets (hence H is a regular G_δ-set).

It is easy to see from Fact 1 that $\{\pi^{-1}(B)\times\pi^{-1}(B') : (\pi^{-1}(B)\times\pi^{-1}(B'))\cap H = \varnothing, B, B' \in \mathscr{B}\}$ is a clopen cover of

$$\{(x^e, y^{e'}) \in X^2\backslash\Delta : (\pi^{-1}(x) \times \pi^{-1}(y)) \cap \overline{H} = \varnothing\}.$$

By Fact 10, the set

$$R = \{(x^e, y^{e'}) \in (X^2\backslash\Delta)\backslash H : (\pi^{-1}(x) \times \pi^{-1}(y)) \cap \overline{H} \neq \varnothing\}$$

is contained in $(X\times E)\cup(E\times X)$ for some countable $E \subset X$. Since $(\{e\}\times X)\backslash H$ and $(X \times \{e\})\backslash H$ are σ-compact (by Fact 9) for each $e \in E$, R can be covered by countably many clopen sets missing H. Thus Fact 11 follows.

Note that we have not yet used the fact that only points of a Luzin set L were split—no matter what uncountable set of points is split, X will be a nonmetrizable space satisfying Facts 1–11.

Fact 12. X^2 is hereditarily normal.

Proof of Fact 12. Suppose $H, K \subset X^2$ are mutually separated, i.e., $\overline{H} \cap K = H \cap \overline{K} = \varnothing$. We will show that K can be covered by countably many open sets whose closures miss H.

Consider the collection

$$\{\pi^{-1}(B_0) \times \pi^{-1}(B_1) : B_i \in \mathscr{B}, B_0 \cap B_1 = \varnothing\}.$$

Each member of this collection is a compact clopen subset of $X^2\backslash\Delta$, hence is hereditarily Lindelöf. Thus the set $\{(x^e, y^{e'}) \in K : x \neq y\}$ can be covered by countably many open sets whose closures miss H.

Further, since X is hereditarily Lindelöf, $K\backslash\Delta$ can be so covered. Also, $K \cap \{(x^e, x^{e'}) : (\pi^{-1}(x) \times \pi^{-1}(x)) \cap \overline{H} = \varnothing\}$ can be so covered (by sets of the form $(\pi^{-1}(B))^2$).

It remains to show that

$$K_1 = \{(x^e, x^{1-e}) \in K : e \in \{0, 1\}, x \in L, \pi^{-1}(x) \times \pi^{-1}(x) \cap \overline{H} \neq \varnothing\}$$

can be so covered. We may assume $|\pi(K_1)| > \omega$. For each $x \in \pi(K_1)$, there are $g, g', h, h' \in 2$ such that one of the following holds:

(a) $(x^h, x^{h'}) \in \overline{\{(y^g, z^{g'}) \in H: y^g \neq z^{g'}\}} = \overline{H \backslash \Delta}$; or

(b) $(x^h, x^{h'}) \in \overline{\{(y^g, z^{g'}) \in H: y^g = z^{g'}\}} = \overline{H \cap \Delta}$.

By Fact 5, (a) occurs for at most countably many $x \in \pi(K_1)$. Thus (b) occurs for uncountable many $x \in \pi(K_1)$.

For such x, we must have $h = h'$ and $y = z$ (and $g = g'$ if $y = z \in L$).

Let $U = X^2 \backslash \overline{K}$ and let

$$U^* = \{x: \pi^{-1}(x) \times \pi^{-1}(x) \subset U\}.$$

Note that U^* is open in C. Suppose $(x^h, x^{h'}) \in \overline{H \cap \Delta}$ and $(x^e, x^{1-e}) \in K$. Then $x \in \overline{U^*} \backslash U^*$, because any open set in X containing a point in Δ contains $\pi^{-1}(B) \times \pi^{-1}(B)$ for some clopen $B \subset C$. But $L \cap (\overline{U^*} \backslash U^*)$ is countable, so there are only countably many x's for which (b) holds. This contradiction completes the proof. □

4. Normality of $X^2 \backslash \Delta$

In this section we show that no axioms beyond ZFC are required to produce an example of a compact space X such that $X^2 \backslash \Delta$ is normal but X is not perfectly normal, answering a question asked of the first author by P. Kombarov in a private communication. Our example is not even hereditarily normal.

Theorem 4.1. *There is a compact separable space X such that $X^2 \backslash \Delta$ is collectionwise normal but X is not hereditarily normal.*

Proof. This example is constructed similar to the example of Theorem 3.1, with the Kunen line technique replaced by van Douwen's technique [vD$_1$]. The underlying set X for our space is $C^0 \cup C^1$, where C^0 and C^1 are disjoint copies of the Cantor set C.

Let D be a nowhere dense subset of C of size continuum c which contains no uncountable closed subset of C. We are going to make D^0 relatively discrete in X.

Let $\{(A_{\alpha,0}, A_{\alpha,1}): \omega \leq \alpha < c\}$ index all pairs of countable subsets of C^2 with

$$|\overline{A}_{\alpha,0} \cap \overline{A}_{\alpha,1} \cap \Delta| > \omega,$$

such that each pair is repeated c times in the indexing.

Let $C = \{x_\alpha: \alpha < \omega_1\}$. We may assume

(i) $A_{\alpha,0} \cup A_{\alpha,1} \subset \{x_\beta: \beta < \alpha\}^2$;

(ii) $(x_\alpha, x_\alpha) \in \overline{A}_{\alpha,0} \cap \overline{A}_{\alpha,1}$;

(iii) if $x_\alpha \in D$, then $(A_{\alpha,0} \cup A_{\alpha,1}) \subset (C \backslash \overline{D})^2$;

(iv) $\{x_n: n \in \omega\}$ is dense in C and misses $\overline{D}$.

Choose sequences $S_{\alpha,i} \subset A_{\alpha,i}$ converging to (x_α, x_α). Also choose sequences $T_{\alpha,i} \subset A_{\alpha,i}$, if they exist, converging to (x_α, x_α) with $x \neq y$ for each $(x, y) \in T_{\alpha,i}$; otherwise let $T_{\alpha,i} = \varnothing$.

Now choose, for $\omega \leq \alpha < \omega_1$ and $\sigma \in 2^2$, disjoint sequences $\{x_\alpha(n, \sigma)\}_{n<\omega} \subset \{x_\beta\}_{\beta<\alpha}$ converging to x_α such that for each $\sigma, \tau \in 2^2$:

(a) $|\{n: \exists y(x_\alpha(n, \sigma), y) \in S_{\alpha,i}\}| = \omega$;

(b) $|\{n: (x_\alpha(n, \sigma), x_\alpha(n+1, \tau)) \in T_{\alpha,i}\}| = \omega$ if $T_{\alpha,i} \neq \varnothing$;

(c) if $x_\alpha \in D$, then $x_\alpha(n, \sigma) \notin \overline{D}$.

Use the $x_\alpha(n, \sigma)$'s to define a bases at x_α^0 and x_α^1 just as in the proof of

Theorem 3.1, with the further stipulation that if $x_\alpha \in D$, then the $B(n, \sigma, e)$'s miss $\overline{D}$. The resulting space X is clearly a compact Hausdorff space.

Fact 1. X is separable; indeed, $R = \{x_n^e : n \in \omega, e \in 2\}$ is dense in X.

Proof. By (iv) above and an easy induction, $x_\alpha^e \in \overline{R}$ for each $\alpha < \omega_1$ and $e \in 2$.

Fact 2. X is not hereditarily normal.

Proof. By the construction, if $x_\alpha \in D$ then basic neighborhoods of x_α^0 miss D^0. Thus D^0 is a closed discrete set of size continuum in the separable space $D^0 \cup R$ (recall $\{x_n\}_{n<\omega} \cap \overline{D} = \varnothing$) and so by Jones' lemma this subspace cannot be normal.

Fact 3. For each $\sigma \in 2^2$, $\{x_\alpha(n, \sigma)^e\}_{n<\omega} \to x_\alpha^{\sigma(e)}$.

Proof. Same as the proof of Fact 3(a) in the proof of Theorem 3.1.

Fact 4. If $e, e', f, f', i < 2$, and either $e = e'$, $f \neq f'$, or $T_{\alpha,i} \neq \varnothing$, then

$$(x_\alpha^e, x_\alpha^{e'}) \in \overline{\{(x^f, y^{f'}) : (x, y) \in A_{\alpha,i}\}}.$$

Proof. If $T_{\alpha,i} \neq \varnothing$, choose $\sigma, \tau \in 2^2$ with $\sigma(f) = e$, $\tau(f') = e'$. By condition (b) above,

$$A = \{n : (x_\alpha(n, \sigma), x_\alpha(n+1, \tau)) \in T_{\alpha,i}\}$$

is infinite. By Fact 3, $\{x_\alpha(n, \sigma)^f, x_\alpha(n+1), \tau)^{f'}\}_{n\in A}$ converges to $(x_\alpha^e, x_\alpha^{e'})$.

Now assume $T_{\alpha,i} = \varnothing$. Then $x = y$ for all but finitely many $(x, y) \in S_{\alpha,i}$. If $f \neq f'$ or $e = e'$, we can choose $\sigma \in 2^2$ with $\sigma(f) = e$, $\sigma(f') = e'$. Then by (a) above,

$$B = \{n : (x_\alpha(n, \sigma), x_\alpha(n, \sigma)) \in S_{\alpha,i}\}$$

is infinite, so $\{(x_\alpha(n, \sigma)^f, x_\alpha(n, \sigma)^{f'})\}_{n\in B}$ converges to $(x_\alpha^e, x_\alpha^{e'})$.

Fact 5. $X^2 \backslash \Delta$ is normal.

Proof of Fact 5. Let H and K be disjoint closed sets in $X^2 \backslash \Delta$. We show that H can be covered by countably many open sets whose closures miss K; then the same is true for K with respect to H, so H and K can be separated by disjoint open sets.

Let $\mathscr{B}$ be a countable clopen base for C, and let $\pi : X \to C$ be the natural projection. Note that for each pair B, B' of disjoint elements of $\mathscr{B}$, $\pi^{-1}(B) \times \pi^{-1}(B')$ is a clopen compact (hence normal) subset of $X^2 \backslash \Delta$. It follows that the set

$$\{(x^e, y^{e'}) \in H : x \neq y\}$$

can be covered by countably many open sets whose closures miss K.

If $(x^e, x^{e'})$ is such that $\pi^{-1}(x) \times \pi^{-1}(x)$ misses the closure $\overline{K}$ of K in X^2, then there is a clopen $B \in \mathscr{B}$ containing x such that $\pi^{-1}(B)$ misses K; so these points can be so covered as well.

Let $E = \{x : \pi^{-1}(x) \times \pi^{-1}(x)$ meets both H and $\overline{K}\}$. Suppose $|E| > \omega$. For each $x \in E$, choose $f, f' \in \{0, 1\}$ such that $(x^f, x^{f'}) \in H$, and choose $e, e', g, g' \in \{0, 1\}$ such that

$$(x^e, x^{e'}) \in \overline{\{(x^g, y^{g'}) \in K\}}.$$

Let E_1 be an uncountable subset of E on which e, e', f, f', g, and g' are fixed. Let A_0 be a countable dense subset of $\{(x, x): x \in E_1\}$ and let A_1 be countable dense in $\{(x, y): (x^g, y^{g'}) \in K\}$. Then $(A_0, A_1) = (A_{\alpha,0}, A_{\alpha,1})$ for some $\alpha < \underline{c}$, so by Fact 4,

$$(x_\alpha^0, x_\alpha^1) \in \overline{\{(x^f, x^{f'}): x \in A_0\}} \cap \overline{\{(x^g, y^{g'}): (x, y) \in A_1\}} \subset H \cap K.$$

(Note that as $(x^f, x^{f'}) \in H \subset X^2 \backslash \Delta$, we have $f \neq f'$; also if $g = g'$, then $x \neq y$ whenever $(x^g, y^{g'}) \in K$, whence $T_{\alpha,1} \neq \varnothing$.) Thus H and K are not disjoint closed subsets of $X^2 \backslash \Delta$; this contradiction completes the proof of Fact 5.

Fact 6. Every closed discrete subspace of $X^2 \backslash \Delta$ is countable.

Proof. Suppose $Y \subset X^2 \backslash \Delta$ is uncountable and closed discrete. Without loss of generality, there exist $f, f' < 2$ such that $(x^e, y^{e'}) \in Y$ implies $e = f$ and $e' = f'$.

As X^2 is compact and first-countable, the closure $\overline{Y}$ of Y in X^2 has uncountable intersection with the diagonal Δ_Y of Y. Thus the closure of $Z = \{(x, y) \in C^2: (x^f, y^{f'}) \in Y\}$ has uncountable intersection with Δ_C. Let $A \subset Z$ be countable and dense. Then $(A, A) = (A_{\alpha,o}, A_{\alpha,1})$ for some α. By Fact 4,

$$(x_\alpha^0, x_\alpha^1) \in \{(x^f, y^{f'}): (x, y) \in A\} \subset \overline{Y} \cap (X^2 \cap \Delta),$$

a contradiction.

Fact 7. $X^2 \backslash \Delta$ is CWN.

Proof. Immediate from Fact 6 and the fact that normal spaces are "countably CWN" (see [E, Theorem 2.1.14]).

5. Sufficient axioms

In this section we show that the existence of an uncountable set $A \subset [0, 1]$ whose square is both a Q-set and a λ'-set (as required in 2.2) follows just from the existence of a Q-set. We also show that "A^2 is a Q-set" does not necessarily imply that $D(A)^2$ is hereditarily normal. For convenience, we recall the following definitions.

Definition. A subset S of a separable metric space X is a *Q-set* [*resp. λ-set*] if every [resp. every countable] subset of S is a G_δ-set in the relative topology of S. A *Q'-set* [*resp. λ'-set*] in X is a subset S of X such that $S \cup Z$ is a Q-set [*resp. λ-set*] for all countable $Z \subset X$.

Note that being a Q-set is a topological property, unlike being a λ'-set, which depends on how a subspace sits in a larger space. But if it is clear from context what the larger space X is, it is customary to omit the phrase "in X".

Lemma 5.1. *Let S be a subset of a separable metric space X. The following are equivalent.*

(i) *S^2 is a Q'-set (in X^2).*
(ii) *S^2 is a Q-set and a λ'-set.*
(iii) *S^2 is a Q-set and S is a λ'-set.*

(iv) $(S \cup N)^2$ *is a* Q'*-set for every countable* $N \subset X$.

Proof. Obviously, (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The square of a λ'-set is a λ'-set (a quick proof appears in [M]), so (iii) $\Rightarrow$ (ii). Next, it is true in general that a Q-set which is a λ'-set in X is a Q'-set in X. To see this, suppose S is a Q-set and a λ'-set in X, and Z is a countable subset of X. It is clearly enough to show A is an F_σ-subspace of $S \cup Z$ whenever $A \subset S$. Now S is a countable union of closed subsets F_n of $S \cup Z$. Let $A_n = A \cap F_n$ and let A_n be the countable union of sets F_{nk} relatively closed in S. But since $F_{nk} \subset F_n$, F_{nk} is closed in $S \cup Z$ as well, and so A is a countable union of closed subspaces of $S \subset Z$.

Thus (ii) $\Rightarrow$ (i). Finally, to show (i) $\Rightarrow$ (iv), it is enough to show $(S \cup N)^2$ is a Q-set for every countable $N \subset X$, since any countable added set is a subset of $(S \cup N \cup N')^2$ for some countable N'. Now $S \times N$ is a G_δ-subset of $S \times (S \cup N)$, which in turn is an F_σ-subset of $(S \cup N)^2$. By imitating the proof of (ii) $\Rightarrow$ (i), we get every subset of $S \times S$ to be F_σ in $S \times (S \cup N)$, and since $S \times \{n\}$ is a Q-set closed in $S \times (S \cup N)$ for each $n \in N$, every subset of $S \times N$ is an F_σ in $S \times (S \cup N)$, which is thus a Q-set. A repetition of the argument, with $S \times (S \cup N)$ in place of $S \times S$, $(S \cup N)^2$ in place of $S \times (S \cup N)$, and $N \times (S \cup N)$ in place of $S \times N$, and coordinates switched where appropriate, establishes that $(S \cup N)^2$ is a Q-set. □

Now, not every Q-set is a Q'-set. There is even a (consistent) example [FM] of an uncountable Q-set S which is concentrated on the rationals, i.e. each relatively closed subset of $S \cup \mathbb{Q}$ disjoint from $\mathbb{Q}$ is countable. Also, [F] it is consistent that there is a Q-set whose square is not a Q-set.

But we can say the following:

Theorem 5.2. *The following axioms are equivalent:*

(I) *There is a* Q*-set.*

(II) *There is a* Q*-set whose square is a* Q*-set.*

(III) *There is a subset of the Cantor set whose square is both a* Q*-set and a* λ'*-set.*

(IV) *There is a dense subset of* $I = [0, 1]$ *whose square is a* Q'*-set.*

Proof. (I) $\Rightarrow$ (II) was shown by Przymusinski [P_2]; see also [M]. That (III) $\Rightarrow$ (IV) follows quickly from Lemma 5.1, (iii) $\to$ (iv), by letting N be a countable dense subset of I. Note that if S is a λ'-set in the Cantor set, it is also a λ'-set in I because the Cantor set is closed, hence a G_δ, in I.

It remains to show (II) $\Rightarrow$ (III). Let X be a Q-set of size ω_1 whose square is a Q-set. A result essentially due to Kuratowski (see [M, Theorem 9.3]) says that any set of reals of size ω_1 is the one-to-one continuous image of a λ'-set. So let X be the one-to-one continuous image of the λ'-set Y. Then Y^2 is both a Q-set and a λ'-set. □

In a letter to the second author, A. Miller observes that if there is an uncountable Q-set, then there is one whose square is a Q-set but *not* a λ'-set. We can show then that there is an uncountable Q-set $A \subset [0, 1]$ such that A^2 is a Q-set, but $D(A)$ is *not* hereditarily normal.

Theorem 5.4. *If there is a* Q*-set* $A \subset [0, 1]$ *such that* A^2 *is a* Q*-set but not a* λ'*-set, then there is a* Q*-set* A' *with the same properties such that* $D(A')^2$ *is not hereditarily normal.*

Proof. Let A satisfy the given conditions. Since the square of a λ'-set is a λ'-set [M], A is not λ'. Let $Z \subset [0, 1]$ be a countable set disjoint from A such that Z is not a G_δ-set in $A \cup Z$. Without loss of generality, we can assume that A is uncountably dense-in-itself, and that $\overline{A \cup Z} = \overline{Z} = \overline{A}$. Since $|A \cup Z| < 2^\omega$, $A \cup Z$ is homeomorphic to a subspace of the Cantor set C. The closure of this subspace is itself homeomorphic to C, so $A \cup Z$ is homeomorphic to a dense subspace of C. Let L be the set of the left-hand endpoints of the deleted middle-thirds. Since the Cantor set is countable dense homogeneous [B] (i.e., given any two countable dense subsets of C, there is an autohomeomorphism of C taking one to the other), $A \cup Z$ is homeomorphic to $A' \cup L$, where A' is homeomorphic to A and disjoint from L. We can assume A' is disjoint from the right endpoints of deleted intervals as well by subtracting a countable subset if necessary.

We will show that $D(A')^2$ is not hereditarily normal. For each $l \in L$, let $r(l)$ be the right-hand endpoint of the deleted interval with left endpoint l. Let

$$H = \{\langle l, r(l)\rangle : l \in L\},$$

and

$$K = \{\langle a^-, a^+\rangle : a \in A'\}.$$

It is easy to check that $\overline{H} \cap K = \varnothing = H \cap \overline{K}$. Consider an open set U containing K. For each positive integer n, let

$$A_n = \{a \in A' : (a^- - 1/n, a^-] \times [a^+, a^+ + 1/n) \subset U\}.$$

Of course, $A' = \bigcup\{A_n : n \in \omega\}$. For some n, $\overline{A}_n \cap L$ must be infinite. Pick $l \in \overline{A}_n \cap L$ such that $r(l) - l < 1/n$. It follows that $\langle l, r(l)\rangle \in \overline{U}$. Thus the closure of every open set containing K meets H, so H and K cannot be put into disjoint open sets. Thus $D(A')^2$ is not hereditarily normal. □

6. Other axiom systems

In the spirit of $\mathfrak{p}$ and the other cardinal numbers in [vD$_2$], we define

$$\mathfrak{q} = \min\{k \mid \text{no set of real numbers of size } k \text{ is a } Q\text{-set}\}.$$

Thus "$\mathfrak{q} > \omega_1$" means "there is an uncountable Q-set".

In [T] a list of eleven equivalent axioms P_i is given, one of which, P_4, is that $\mathfrak{q} > \omega_1$. Another is:

P_8. There is a separable, normal, first countable space with an uncountable closed discrete subspace.

Let $A \subset [0, 1]$ be uncountable, and A^- and A^+ the obvious subspaces of $D(A)$. Then $(A^- \times A^+)$ satisfies all these conditions except perhaps normality; in particular, $\{\langle a^-, a^+\rangle : a \in A\}$ is a closed discrete subspace. So, if it is normal, P_8 holds and $\mathfrak{q} > \omega_1$ also.

The following is folklore:

Lemma 6.1. *The axiom* P_8 *is equivalent to*:

P'_8. *There is a separable, hereditarily normal, first countable space with an uncountable discrete subspace.*

Proof. If P_8, then there is a separable, normal nonmetrizable Moore space [T]. In such a space there must be an uncountable closed discrete subspace [J]. But in a Moore space, every closed subset is a G_δ, so that perfect normality and hence hereditarily normality follows.

If P'_8, let Q be a countable dense subspace and D an uncountable discrete subspace in a witnessing space. Let $Q' = Q \cap \overline{D}$ and let D' be a countable subset of D with Q' in its closure. Then D' does not have any other points of D in its closure, and so neither does Q'. Therefore $Q \backslash Q'$ is a dense subspace of $(Q \backslash Q') \cup (D \backslash D')$, and $D \backslash D'$ is a closed discrete subspace. □

Another axiom equivalent to $\mathfrak{q} > \omega_1$ is the existence of an uncountable $A \subset [0, 1]$ such that $(D(A))^2$ is hereditarily normal (Theorems 2.2 and 5.2). See also Theorem 6.2 and the comments following Theorem 6.7, below.

A classic result of F. B. Jones [J], also treated in [T and Ny_2], is that P_8 and P'_8 imply $2^\omega = 2^{\omega_1}$. So it is natural to ask what happens to Katětov's problem if $\mathfrak{q} = \omega_1$ and in particular if $2^\omega < 2^{\omega_1}$.

Linearly ordered spaces can be ruled out for Katětov's problem if $\mathfrak{q} = \omega_1$. This is a result of Zenor (see $[\text{L}_1]$), which together with 2.2 and 5.2 gives

Theorem 6.2. *The axiom "there is a linearly orderable, nonmetrizable compact space whose square is hereditarily normal" is equivalent to* $\mathfrak{q} > \omega_1$.

Zenor's argument for $\Rightarrow$ depends on a slightly incomplete argument in [Ru], so we here sketch the former and complete the latter.

A compact perfectly normal space is hereditarily Lindelöf [E, Exercise 3.8.A] and first countable (because each point is a G_δ). If it is linearly orderable, it is separable [Ru]. If it is not metrizable, it has uncountably many pairs $\{b^-, b^+\}$ with b^- the immediate predecessor of b^+, and these behave like A^- and A^+ in the discussion following the statement of P_8. So hereditary normality of the square implies $\mathfrak{q} > \omega_1$.

The argument for separability in [Ru] was carried out for the connected case, in the form: a compact Souslin line (i.e. a connected, linearly orderable, compact, perfectly normal, nonseparable space) cannot have a hereditarily normal square. In [Ru] it is claimed that the argument carries over verbatim even if connectedness is dropped. However, in this more general case one must explicitly require that the set of all immediate successors of each element of the tree T of intervals constructed there should have the order type of the rationals in the natural order, and that the lower endpoint of each interval should not have an immediate successor point, nor should the upper endpoint have an immediate predecessor. [These are automatic in the connected case, and easily arranged in general.] The argument in [Ru] actually establishes that $X^2 \backslash \Delta$ is not hereditarily normal.

The intrusion of Souslin spaces is a special case of the way S and L spaces come into the picture whenever $\mathfrak{q} = \omega_1$.

Definition 6.3. An *S-space* is a hereditarily separable regular space that is not hereditarily Lindelöf. An *L-space* is a hereditarily Lindelöf regular space that is not hereditarily separable. A space is of *countable spread* if every discrete subspace is countable.

Of course, every hereditarily separable space is of countable spread, and so is every hereditarily Lindelöf space. The following oft-used theorem strengthens the connection between these concepts.

Theorem E [Ro, Theorems 3.1 and 3.3]. *Let X be a regular space of countable spread. Then*:

(1) *If X is not hereditarily Lindelöf, then X contains an S-space, and*

(2) *If X is not hereditarily separable, then X contains an L-space.*

An important theorem of Szentmiklóssy [Sz] is

Theorem F. *If* $\mathrm{MA} + \neg\mathrm{CH}$, *then no subspace of a countable tight* (*in particular, first countable*) *compact space can be an S-space or an L-space.*

Theorem 6.4[1]. *Let X be a compact* (*perfectly normal, hence hereditarily Lindelöf*) *nonmetrizable space such that X^2 is hereditarily normal. If $\mathfrak{q} = \omega_1$, or if X^2 is hereditarily CWH, then at least one of the following is true.*

(1) *X is an L-space.*

(2) *X^2 is an S-space.*

(3) *X^2 contains both an S-space and an L-space.*

On the other hand, if $\mathfrak{q} > \omega_1$, all three can fail.

Proof. If X is not an L-space, it is separable and hence so is X^2. If X^2 is hereditarily CWH, then this implies X^2 is of countable spread. We can also conclude this if $\mathfrak{q} = \omega_1$ because P_8' is negated. Since X^2 is not perfectly normal by Šneĭder's theorem, it is not hereditarily Lindelöf. By Theorem F, X^2 then contains an S-space, and if X^2 is not itself an S-space, it also contains an L-space. □

If $\mathfrak{q} > \omega_1$, then the $D(A)$ of Theorem 2.2 satisfies none of the conditions (1)–(3). This will be shown in Corollary 6.10 below.

Corollary 6.5. $(\mathrm{MA} + \neg\mathrm{CH})$. *A compact space X is metrizable if and only if X^2 is hereditarily normal and hereditarily CWH.*

Proof. Sufficiency follows from 6.4 and Theorem F. Necessity is clear. □

We can drop the hereditary normality assumption in 6.5 if we assume X is hereditarily *strongly CWH* (i.e., discrete collections of points can be separated by *discrete* collections of open sets).

For this we use the following lemma, reminiscent of Theorem 1 of [K]. Recall that a space is called *ω_1-compact* if every closed discrete subspace is countable; equivalently, every uncountable subset has a limit point.

Lemma 6.6. *If X is ω_1-compact and $X \times Y$ is hereditarily strongly CWH, then either X is of countable spread, or every countable subset of Y is closed discrete.*

Proof. Suppose on the contrary that $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable discrete subset of X while p is a limit point of $\{y_n : n \in \omega\} \subset Y$. Then $\{(x_\alpha, p) : \alpha < \omega_1\}$ is closed discrete in

$$Z = [X \times (\{y_n\}_{n\in\omega} \cup \{p\})] \backslash [X \backslash \{x_\alpha\}_{\alpha<\omega_1} \times \{p\}].$$

If $(x_\alpha, p) \in U_\alpha \times V_\alpha$, $\alpha < \omega_1$, where U_α and V_α are open, then for some uncountable $A \subset \omega_1$ and some $n \in \omega$, we have $y_n \in V_\alpha$ for all $\alpha \in A$. Choose a limit point q of $\{x_\alpha\}_{\alpha \in A}$; then $(q, y_n) \in Z$ is a limit point of the $U_\alpha \times V_\alpha$'s. It follows that Z is not strongly CWH, a contradiction. □

Theorem 6.7. $(\mathrm{MA} + \neg\mathrm{CH})$. *Let X and Y be compact.*

[1]Part of this proof follows the proof, in [Ny_2], of the weaker theorem at the end of [Ny_1].

(a) *$X \times Y$ is hereditarily strongly CWH iff $X \times Y$ is hereditarily Lindelöf iff $X \times Y$ is perfectly normal.*

(b) *X^2 is hereditarily strongly CWH iff X is metrizable.*

Proof. (a) By 6.6, both X and Y have countable spread if $X \times Y$ is hereditarily strongly CWH. Now tightness $\leq$ spread for compact spaces [A], so both have countable tightness. Theorems E and F (2) then imply they are separable. But now $X \times Y$ is also separable, and being hereditarily CWH, it is also of countable spread. By Theorems E and F (1), $X \times Y$ is hereditarily Lindelöf. The remaining implications follow from the fact that a regular space is hereditarily Lindelöf iff it is perfectly normal and Lindelöf [E, Exercise 3.8.A] and that every hereditarily Lindelöf space is hereditarily normal and hereditarily ω_1-compact, hence hereditarily strongly CWH.

(b) Follows from (a) and Šneĭder's theorem. □

Since Example 3.1 under CH has X^2 hereditarily separable and hereditarily normal, hence hereditarily CWN, we obtain three statements independent of ZFC:

If X is compact and X^2 is one of {hereditarily CWN, hereditarily normal and hereditarily CWH, hereditarily strongly CWH}, then X is metrizable.

We do not know if "hereditarily CWH" can be added to the list:

Problem. Does $MA + \neg CH$ imply a compact space with hereditarily CWH square is metrizable?

We can weaken the topological hypotheses in Theorem 6.7 in other directions if we assume the Proper Forcing Axiom (PFA), a strengthening of $MA + \neg CH$.

Theorem 6.8. (PFA) *Let X and Y be infinite, ω_1-compact, regular spaces.*

(a) *If $X \times Y$ is hereditarily strongly CWH, then both X and Y are hereditarily Lindelöf.*

(b) *If X and Y are countably compact, the following are equivalent.*

  (i) *$X \times Y$ is hereditarily strongly CWH.*

  (ii) *$X \times Y$ is hereditarily Lindelöf.*

  (iii) *$X \times Y$ is perfectly normal.*

(c) *If X is countably compact then X^2 is hereditarily strongly CWH iff X is metrizable.*

Proof. (a) By 6.6, both X and Y are of countable spread. The PFA implies there are no S-spaces [To_1], so the result follows by Theorem F.

(b) Each of (i), (ii), and (iii) implies X and Y are compact. For (i) and (ii) use (a) and the obvious ZFC fact that Lindelöf + countably compact $\Rightarrow$ compact, while for (iii) use Weiss's Theorem [W_2] that $MA + \neg CH$ implies countably compact, perfectly normal spaces are compact. So (b) follows from 6.7.

(c) By (a) and countable compactness, X is compact and so we use 6.7. □

The following theorem helps justify the final assertion in Theorem 6.4.

Theorem 6.9. *Let X be a subspace of a finite product of spaces, each homeomorphic to either the Sorgenfrey line S or the real line R. The following are equivalent:*

(i) *X is ω_1-compact.*

(ii) X *is of countable spread.*
(iii) X *is hereditarily Lindelöf.*
(iv) X *is hereditarily separable.*

Proof. (iv) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (i) are clearly true for all spaces. We will show (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii), which will complete the proof.

We show (iii) $\Rightarrow$ (iv) for $X \subset S^n \times R^m$ by induction on N. It is obvious if $n = 0$ because then X is second countable. Assume true for $n-1$ and let $Y \subset X$. Let A be a countable subset of Y that is dense in the coarser (Euclidean) topology of Y.

If $Y \backslash \overline{A} \neq \varnothing$, then for each $y \in Y \backslash \overline{A}$ let $B_{m(y)}(y)$ be a basic open neighborhood of y missing $\overline{A}$. Note that any point of $B_{m(y)}(y)$ which disagrees with y in every coordinate is in the Euclidean interior of $B_{m(y)}(y)$ and hence cannot be in $Y \backslash \overline{A}$. Let $\{y_i \colon i \in \omega\}$ be such that these neighborhoods of the y_i cover $Y \backslash \overline{A}$. Let

$$D_{j,i} = \{x \in Y \backslash \overline{A} \colon \text{ the } j\text{th coordinate of } x \text{ and } y_i \text{ are equal}\}.$$

Then $Y \backslash \overline{A} = \bigcup\{D_{j,i} \colon i \in \omega \text{ and } j \leq n\}$, and each $D_{j,i}$ is homeomorphic to a subset of $S^{n-1} \times R^m$. Thus, $D_{j,i}$ has a countable dense subspace by induction, and so does Y.

To show (i) $\Rightarrow$ (iii), we use the fact that $S^n \times R^m$ is perfect [HM] and subparacompact [L] for finite n, m. [A space is *perfect* if every closed subspace is a G_δ, and *subparacompact* if every open cover has a σ-discrete closed refinement.] Then X is also subparacompact by the following lemma, which ought to have appeared somewhere a long time ago, but we could find no reference to it:

Lemma G. *Every perfect, subparacompact space is hereditarily subparacompact.*

Once this is proven, all that remains is the elementary observation that every subparacompact, ω_1-compact space is Lindelöf.

Proof of Lemma G. Let X be as in the hypothesis. First we show every open subspace is subparacompact. Let U be an open subspace, and let $U = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed. Let $\mathscr{W}$ be an open cover of U and let W_n be its trace on F_n, i.e. $W_n = \{W \cap F_n \colon W \in \mathscr{W}\}$. Of course, every closed subspace of a subparacompact space is subparacompact, so each W_n has a σ-discrete closed refinement $\mathscr{K}_n$ covering F_n. But $\mathscr{K}_n$ is both σ-discrete and closed in X as well, and hence in U, so the union of all the $\mathscr{K}_n$ is the desired refinement of W.

Now let Y be any subspace of X, and let $\mathscr{V}$ be a relatively open cover of Y. Replace each V in $\mathscr{V}$ by an open subset W of X such that $V = W \cap Y$, and let U be the union of all the resulting W. Of course, $Y \subset U$. Take a σ-discrete (in U) refinement $\mathscr{K}$ of the W's by closed (in U) subsets of U. Then $\{K \cap Y \colon K \in \mathscr{K}\}$ is the desired refinement of $\mathscr{V}$. □

Corollary 6.10. *Let X be formed by splitting points in a subset A of R. No finite power of X contains an S-space or an L-space.*

Proof. It is easy to see that any finite power of X is a finite union of subspaces of the form $S^n \times R^m$. Now use the obvious fact that the finite union of hereditarily separable spaces is hereditarily separable, and similarly for hereditarily Lindelöf spaces. □

Theorem 6.4 is significant for another reason. There are no known constructions of S-spaces or L-spaces using only $2^\omega < 2^{\omega_1}$, let alone ones using $\mathfrak{q} = \omega_1$ that can be embedded in compact first countable spaces. So we say:

Conjecture. It is consistent that every compact space with a hereditarily normal square is metrizable.

To vindicate this, it may be necessary to come up with new ways of negating the existence of various kinds of S- and L-spaces. As a careful reading of [R_0] reveals, the precise set-theoretic hypothesis that makes Szentmiklóssy's theorem go through is

Axiom K. Every poset P such that P^n is ccc for all finite n satisfies Property K.

Property K says that for every uncountable subset S of P there is an uncountable $S' \subset S$ such that any two elements of S' are compatible in P. Now it is not known whether Axiom K is compatible with $2^\omega < 2^{\omega_1}$, although Todorcevic has recently shown that its strengthening with "two" replaced by "three" in the statement of Property K is not. So it seems unlikely that Axiom K is compatible with $2^\omega < 2^{\omega_1}$, although we may have better luck with $\mathfrak{q} = \omega_1$. But perhaps the best route is to find new ways of destroying S- and L-subspaces of compact, first countable, hereditarily normal spaces.

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